5. Bautin, N. N., Qualitative investigation of a dynamic system. PMM Vol. 36, №3, 1972.
6. Tricomi, F., Integrazione di unequazione differenziale presentatasi in elettrotechnica. Ann. Scuola norm. super. Pisa Sci. fis. e mat. Ser. 11, Vol. 2, 1933.
7. Dulac, H., Recherche des cycles limités. C. r. Acad. sci., t. 204, №23, 1937.

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CASE OF A GENERATING FAMILY OF QUASI-PERIODIC SOLUTIONS IN THE THEORY OF SMALL PARAMETER

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R. F. NAGAEV
(Leningrad)
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We consider the problem of the existence and the stability in-the-small of periodic solutions of systems of ordinary differential equations with a small parameter μ, which in the generating approximation ($\mu = 0$) admit of a family of quasi-periodic solutions (we are concerned only with the solutions belonging to the indicated family when $\mu = 0$). The case to be investigated is in a specific sense a more general case of the unisolated generating solution in the small parameter theory and, therefore, includes everything previously treated by Malkin [1], Blekhman [2], and others. The main difficulty in the investigation is the presence of a multiple zero root in the characteristic determinant of the problem's generating system, to which both simple as well as quadratic elementary divisors [3] correspond. This fact predestines the presence of three groups of stability criteria for the solution being examined. The method for constructing these criteria, proposed here, assumes, in contrast to a previous one [1], the preliminary determination of not only the generating approximation but also the first one to the desired periodic solution. Particular aspects of the general "mixed" problem treated here were studied earlier in [4, 5].

1. Existence of a periodic solution. At present relatively general integrability tests and integration methods for systems of high-order nonlinear differential equations have been worked out only for autonomous canonical systems [6]. The successive use of these methods leads, in the case of a sign-definite Hamiltonian function, to the determination of a general quasi-periodic integral. The conjugate canonic variables of the problem are here expressed as 2π-periodic functions of the quantities

$$\psi_s = \nu_s t + \alpha_s \tag{1.1}$$

and also of the mutually independent integration constants h_s $(s = 1, 2, \ldots)$. Naturally, the total number of quantities ψ_s, h_s equals the order of the original system. If the quantities α_s are integration constants, also independent of each other and of h_s (and we assume this is so in what follows), then the quantities ψ_s acquire the nature of partial rapidly-rotating phases and, moreover, the partial frequencies ν_s depend, as does the energy constant, only on the h_s in the general case. Sometimes the constants h_s take on the meaning of partial actions. In that case we usually speak of the pair h_s, ψ_s as the canonic pair "action — angle".

A more common case can occur when the system under investigation can be separated into two subsystems by some means or other. One of them being canonical and integrable, the other, either linear and stationary relative to the proper variables or admitting of the construction of a particular periodic solution by Liapunov's method [1] or by some other local method. Here, naturally, the family of quasi-periodic solutions of the original system can be constructed by purely analytic means. Thus, the periodic solution, constructed by known analytic methods, of a high-order system is in practice always a part of some family of quasi-periodic solutions.

Keeping what has been said in mind, in the general case we assume that the system

$$x = X(x, \psi, \mu) \tag{1.2}$$

where x is a $k \times 1$ vector, the vector-valued function X is analytic in x and μ and 2π-periodic in the dimensionless argument $\psi = \nu t$ $(\nu > 0)$, admits of a family of quasi-periodic solution for $\mu = 0$

$$x(t, 0) = \varphi(\psi, \psi_1, \ldots, \psi_l, h_1, \ldots, h_n) \tag{1.3}$$

Here the rapidly rotating phases ψ_s $(s = 1, \ldots, l)$ and the constants h_s $(s = 1, \ldots, n)$ have the previous meaning. Furthermore, $l + n \leqslant k$. For the sake of generality we assume that the first m partial frequencies of family (1.3) essentially depend on the constants $h_1, \ldots, h_n$, while the $l - m$ succeeding ones identically equal ν, i.e.

$$\begin{aligned} \nu_s &= \nu_s(h_1, \ldots, h_n) \qquad (s = 1, \ldots, m) \\ \nu_s &\equiv \nu \qquad\qquad\qquad\quad (s = m + 1, \ldots, l) \end{aligned} \tag{1.4}$$

We note also that the Fourier expansion of the vector-valued function φ may, in general, not contain several (but not very many) first harmonics of the phases $\psi, \psi_1, \ldots, \psi_l$.

The membership of (1.3) in the subfamily of solutions T-periodic in t $(T = 2\pi / \nu)$ is characterized by the fulfillment of the relations

$$\nu_s(h_1, \ldots, h_n) = \nu \qquad (s = 1, \ldots, m) \tag{1.5}$$

and, therefore, depends only on $l + n - m$ independent constants among which are the phase shifts $\alpha_1, \ldots, \alpha_l$. Therefore, the variational system

$$y^{\cdot} = \frac{\partial X}{\partial x} y \tag{1.6}$$

which for $\mu = 0$ $(y = y_0)$ is not far from the above-mentioned periodic motions, admits, in accordance to a theorem of Poincaré [1], of l mutually independent T-periodic solutions $\partial\varphi / \partial\alpha_i$ $(i = 1, \ldots, l)$ and of n linearly increasing

$$\frac{\partial'\varphi}{\partial h_s} = \sum_{i=1}^{m} \frac{\partial\varphi}{\partial\alpha_i}\frac{\partial\nu_i}{\partial h_s} t + \frac{\partial\varphi}{\partial h_s} \qquad (s = 1, \ldots, n) \tag{1.7}$$

The prime here denotes "total" partial differentiation.

The family (1.7) of n mutually independent solutions of the linear system (1.6) can be replaced when $\mu = 0$ by a family of m linearly increasing and $n - m$ periodic solutions. Indeed, under condition (1.5) we assume that the determinant

$$\left|\frac{\partial \nu_i}{\partial h_s}\right|_{i,\, s=1,\ldots,m}$$

is nonzero. Then, the numbers χ_{sr} are completely uniquely determined from the relations

$$\sum_{s=1}^{m} \chi_{sr} \frac{\partial \nu_i}{\partial h_s} = \delta_{ir} \qquad (i, r = 1, \ldots, m) \tag{1.8}$$

The quantities

$$\vartheta_r = \sum_{s=1}^{m} \chi_{sr} \frac{\partial' \varphi}{\partial h_s} \tag{1.9}$$

are linear combinations of the first m solutions in (1.7) and, therefore, also satisfy system (1.6) with $\mu = 0$. On the other hand, if we substitute (1.7) into (1.9), then as a consequence of (1.8) we have, after replacing the index r by i,

$$\vartheta_i = \frac{\partial \varphi}{\partial \alpha_i} t + \sigma_i \qquad (i = 1, \ldots, m) \tag{1.10}$$

Here the periodic component σ_i is determined from the formula

$$\sigma_i = \sum_{s=1}^{m} \chi_{si} \frac{\partial \varphi}{\partial h_s} \tag{1.11}$$

and, as a consequence of (1.6) and (1.10), satisfies the equation

$$\sigma_i^{\cdot} = \left(\frac{\partial X}{\partial x}\right) \sigma_i - \frac{\partial \varphi}{\partial \alpha_i} \qquad (i = 1, \ldots, m) \tag{1.12}$$

In Eqs. (1.12) and below the parantheses denote that the corresponding quantity is computed for $x = \varphi$ and $\mu = 0$.

We introduce the quantities

$$\vartheta_s = \frac{\partial' \varphi}{\partial h_s} - \sum_{i,\, r=1}^{m} \chi_{ir} \frac{\partial' \varphi}{\partial h_i} \frac{\partial \nu_r}{\partial h_s} \qquad (s = m+1, \ldots, n) \tag{1.13}$$

which obviously also satisfy system (1.6) with $\mu = 0$. Since by virtue of (1.8) the linearly increasing terms vanish after the substitution of (1.7) into (1.13), the quantities ϑ_s $(s = m + 1, \ldots, n)$ are periodic in t and can be represented as

$$\vartheta_s = \frac{\partial \varphi}{\partial h_s} - \sum_{i,\, r=1}^{m} \chi_{ir} \frac{\partial \varphi}{\partial h_i} \frac{\partial \nu_r}{\partial h_s} \qquad (s = m+1, \ldots, n) \tag{1.14}$$

Together with (1.14) the following expressions, ensuing from (1.11), are also valid:

$$\vartheta_s = \frac{\partial \varphi}{\partial h_s} - \sum_{i=1}^{m} \frac{\partial \nu_i}{\partial h_s} \sigma_i \qquad (s = m+1, \ldots, n) \tag{1.15}$$

Thus, the system of variational equations of the generating system of the problem ($\mu = 0$) admits of $l + n - m$ mutually independent T-periodic solutions $\partial\varphi^{\cdot} / \partial\alpha_i$ $(i = 1, \ldots, l)$ and ϑ_s $(s = m + 1, \ldots, n)$. We assume that this system admits of no other independent T-periodic solutions. Then the conjugate system

$$z^{\cdot} = -z\,(\partial X / \partial x) \tag{1.16}$$

also admits of $l + n - m$ mutually independent T-periodic particular solutions which we subsequently denote by $z_1, \ldots, z_{l+n-m}$.

In connection with the problem being considered, it has been shown in local small parametr theory [1] that for the analyticity in μ and the T-periodicity in t of the solution $x\ (t,\ \alpha_1,\ \ldots,\ \alpha_l,\ h_1,\ \ldots,\ h_n,\ \mu)$ of system (1.2), which becomes (1.3) when $\mu = 0$, it is sufficient that the transcendental system consisting of the m equations in (1.5) and, in addition, of the $l + n - m$ equations

$$P_s(\alpha_1, \ldots, \alpha_l, h_1, \ldots, h_n) \equiv \int_0^T z_s \left(\frac{\partial X}{\partial \mu}\right)^{\cdot} dt = 0 \quad (s = 1, \ldots, l + n - m) \qquad (1.17)$$

admit of simple solutions. These solutions, of course, uniquely determine the T-periodic generating approximation in the expansion

$$x = \varphi + \mu x_1 + \mu^2 x_2 + \mu^3 \ldots$$

On the other hand, the presence of such a solution guarantees the T-periodicity of the sequence of corrections $x_1,\ x_2,\ \ldots$. We note that the first approximation $x_1 = (\partial x / \partial \mu)$ to the solution $x\ (t, \alpha_1,\ \ldots,\ \alpha_l,\ h_1,\ \ldots,\ h_n,\ \mu)$, not necessarily periodic, can be found from the system

$$x_1^{\cdot} = \left(\frac{\partial X}{\partial x}\right) x_1 + \left(\frac{\partial X}{\partial \mu}\right)$$

2. Stability criteria of the first group. The analysis in Sect. 1 shows that the variational equations of the generating system admit of m groups of solutions $\partial\varphi/\partial\alpha_i$, $\vartheta_i\ (i = 1,\ \ldots,\ m)$, to which correspond zero characteristic indices with quadratic elementary divisors [1, 3]. The roots of the characteristic determinant of system (1.6), corresponding to these solutions, are analytic in $\mu^{1/2}$ [7]. On the other hand, to the simple periodic solutions $\partial\varphi / \partial\alpha_i\ (i = m + 1,\ \ldots,\ l)$, $\vartheta_s\ (s = m + 1,\ \ldots,\ n)$ there correspond zero characteristic indices with simple elementary divisors. The characteristic indices of system (1.6), reducing to them as $\mu \to 0$, are analytic in μ.

We turn to the direct determination of the "critical" solutions of system (1.6). Introducing the substitution [1]

$$y = \eta \exp \lambda\ (\mu)\ t \qquad (2.1)$$

we seek the T-periodic solutions of the system

$$\eta^{\cdot} = \frac{\partial X}{\partial x} \eta - \lambda\eta \qquad (2.2)$$

We represent the T-periodic particular solutions, analytic in μ, of (2.2) in the form

$$\eta = \eta_0 + \mu\eta_1 + \mu^2 \ldots, \qquad \lambda = \lambda_1\mu + \mu^2 \ldots \qquad (2.3)$$

Obviously, the generating periodic approximation η_0, can be represented as a superposition of periodic solutions of the variational equations of the generating system, i.e.

$$\eta_0 = \sum_{i=1}^{l} \frac{\partial\varphi}{\partial\alpha_i} a_i + \sum_{s=m+1}^{n} \vartheta_s b_s \qquad (2.4)$$

where $a_1,\ \ldots,\ a_l,\ b_{m+1},\ \ldots,\ b_n$ are constants. Differentiating (2.2) with respect to μ and then setting $\mu = 0$, we obtain and equation for determining η_1

$$\eta_1^{\cdot} = \left(\frac{\partial X}{\partial x}\right) \eta_1 + \left[\left(\frac{\partial'}{\partial\mu}\frac{\partial X}{\partial x}\right) - \lambda_1 E_k\right] \left(\sum_{i=1}^{l} \frac{\partial\varphi}{\partial\alpha_i} a_i + \sum_{s=m+1}^{n} \vartheta_s b_s\right) \qquad (2.5)$$

where E_k is the $k \times k$ unit matrix. Since by virtue of (1.2)

$$\left[\frac{\partial'}{\partial\mu}\frac{\partial X}{\partial x}\right]\frac{\partial x}{\partial\alpha_i}\equiv\frac{\partial^2 x^{\cdot}}{\partial\mu\partial\alpha_i}-\frac{\partial X}{\partial x}\frac{\partial^2 x}{\partial\mu\partial\alpha_i} \tag{2.6}$$

with due regard to (1.13) system (2.5) can be reduced to the form

$$\eta_1^{\cdot}=\left(\frac{\partial X}{\partial x}\right)\eta_1+\sum_{i=1}^{l}\left[\frac{\partial x_1^{\cdot}}{\partial\alpha_i}-\left(\frac{\partial X}{\partial x}\right)\frac{\partial x_1}{\partial\alpha_i}-\lambda_1\frac{\partial\varphi}{\partial\alpha_i}\right]a_i+ \tag{2.7}$$

$$\sum_{s=m+1}^{n}\left[\frac{\partial^* x_1^{\cdot}}{\partial h_s}-\left(\frac{\partial X}{\partial x}\right)\frac{\partial^* x_1}{\partial h_s}-\lambda_1\vartheta_s\right]b_s$$

Here we have introduced the notation

$$\frac{\partial^*}{\partial h_s}=\frac{\partial}{\partial h_s}-\sum_{i,\,r=1}^{m}\chi_{ir}\frac{\partial\nu_r}{\partial h_s}\frac{\partial}{\partial h_i} \tag{2.8}$$

We now carry out a partial orthogonalization and normalization of the periodic solutions of system (1.16) in accordance with the equalities

$$z_r\frac{\partial\varphi}{\partial\alpha_i}=\delta_{ir}\quad(i=m+1,\ldots,l),\qquad z_r\vartheta_s=\delta_{l-m+s,\,r}\quad(s=m+1,\ldots,n) \tag{2.9}$$

Furthermore, since the quantities ϑ_i $(i=1,\ \ldots,\ m)$, defined by (1.10), satisfy the relations $z_r\vartheta_i\equiv\mathrm{const}$ and, consequently,

$$z_r\frac{\partial\varphi}{\partial\alpha_i}\equiv 0\qquad(i=1,\ldots,m) \tag{2.10}$$

after some manipulations with due regard to (1.16) and (2.7), we obtain

$$\frac{d}{dt}z_r\left(\eta_1-\sum_{i=1}^{l}\frac{\partial x_1}{\partial\alpha_i}a_i-\sum_{s=m+1}^{n}\frac{\partial^* x_1}{\partial h_s}b_s\right)=\begin{cases}0\quad(r=1,\ldots,m)\\-\lambda_1 a_r\quad(r=m+1,\ldots,l)\\-\lambda_1 b_{r-l+m}\ (r=l+1,\ldots\\\qquad\ldots,l+n-m)\end{cases} \tag{2.11}$$

We integrate these relations with respect to t in the limits from zero to T. Then, since

$$z_r\frac{\partial x_1}{\partial\alpha_i}\Big|_0^T\equiv\frac{\partial}{\partial\alpha_i}(z_r\bar{x}_1)\Big|_0^T,\qquad z_r\frac{\partial x_1}{\partial h_s}\Big|_0^T\equiv\frac{\partial}{\partial h_s}(z_r x_1)\Big|_0^T,\quad z_r x_1\big|_0^T\equiv P_r \tag{2.12}$$

it turns out that the fulfillment of the following linear equations:

$$\sum_{i=1}^{l}\frac{\partial P_r}{\partial\alpha_i}\alpha_i+\sum_{s=m+1}^{n}\frac{\partial^* P_r}{\partial h_s}b_s=\begin{cases}0 & (r=1,\ldots,m)\\\lambda_1 Ta_r & (r=m+1,\ldots,l)\\\lambda_1 Tb_{r-l+m} & (r=l+1,\ldots,l+n-m)\end{cases} \tag{2.13}$$

is necessary and sufficient for the T-periodicity of the quantity η_1. Equating the determinant of this system to zero, we arrive at an equation of degree $l+n-2m$ for determining the first approximations to the "simple" critical characteristic indices of the problem. The $l+n-2m$ stability criteria following from this we call stability criteria of the first group.

3. Additional stability criteria. In contrast to (2.3) the "nonsimple" critical solutions of system (2.2) are represented as

$$\eta = \eta_0 + \mu^{1/2}\eta_1 + \mu\eta_2 + \mu^{3/2} \ldots, \qquad \lambda = \lambda_1\mu^{1/2} + \lambda_2\mu + \mu^{3/2} \tag{3.1}$$

Here, as before, the periodic generating approximation can be written in form (2.4). However, by virtue of the fact that the components of the matrix $\partial X / \partial x$ are analytic in μ, the system of equations for determining the first approximation η_1 has the form

$$\eta_1^{\cdot} = \left(\frac{\partial X}{\partial x}\right) \eta_1 - \lambda_1 \left(\sum_{i=1}^{l} \frac{\partial \varphi}{\partial \alpha_i} a_i + \sum_{s=m+1}^{n} \vartheta_s b_s \right) \tag{3.2}$$

The general T-periodic solution of this system exists only if $a_{m+1} = \ldots = a_l = b_{m+1} = \ldots = b_n = 0$ and, by virtue of (1.12), can be written in the form

$$\eta_1 = \lambda_1 \sum_{i=1}^{m} \sigma_i a_i + \sum_{i=1}^{l} \frac{\partial \varphi}{\partial \alpha_i} c_i + \sum_{s=m+1}^{n} \vartheta_s d_s \tag{3.3}$$

where $c_1, \ldots, c_l, d_{m+1}, \ldots, d_n$ are the new constants of integration. Thus, η_1 depends now on $l + n$ constants. The periodic second approximation η_2 is determined from the system

$$\eta_2^{\cdot} = \left(\frac{\partial X}{\partial x}\right)\eta_2 + \sum_{i=1}^{m} \left\{\left[\left(\frac{\partial'}{\partial \mu} \frac{\partial X}{\partial x}\right) - \lambda_2 E_k \right]\frac{\partial \varphi}{\partial \alpha_i} - \lambda_1^2 \sigma_i\right\} a_i - \tag{3.4}$$
$$\lambda_1 \left(\sum_{i=1}^{l} \frac{\partial \varphi}{\partial \alpha_i} c_i + \sum_{s=m+1}^{n} \vartheta_s d_s \right)$$

We now assume the fulfillment of the relations

$$z_r \sigma_i = \delta_{ir} \qquad (i = 1, \ldots, m) \tag{3.5}$$

which together with (2.9) completely determine the choice of the orthogonal and normalized solutions $z_1, \ldots, z_{l+n-m}$ of system (1.16). Then, proceeding as before, with due regard to (2.6) and (3.4), we obtain

$$\frac{d}{dt} z_r \left(\eta_2 - \sum_{i=1}^{m} \frac{\partial x_i}{\partial \alpha_i} a_i\right) = \begin{cases} -\lambda_1^2 a_r & (r = 1, \ldots, m) \\ -\lambda_1 c_r & (r = m+1, \ldots, l) \\ -\lambda_1 d_{r-l+m} & (r = l+1, \ldots, l+n-m) \end{cases} \tag{3.6}$$

Integrating these relations with respect to t in the limits from 0 to T, with due regard to (2.11), we get that for the periodicity of η_2 it is necessary and sufficient to fulfill the following $l + n - m$ homogeneous linear equations in the unknowns $a_1, \ldots, a_m, c_{m+1}, \ldots, c_l, d_{m+1}, \ldots, d_n$:

$$\sum_{i=1}^{m} \frac{\partial P_r}{\partial \alpha_i} a_i = \begin{cases} \lambda_1^2 T a_r & (r = 1, \ldots, m) \\ \lambda_1 T c_r & (r = m+1, \ldots, l) \\ \lambda_1 T d_{r-l+m} & (r = l+1, \ldots, l+n-m) \end{cases} \tag{3.7}$$

We note that as a consequence of (1.12) the values of the constants $c_1, \ldots, c_m$ and λ_2 prove to have no influence on the T-periodicity of η_2 The expression for η_2 obtained as a result of integrating (3.4) can be written in the following general form:

$$\eta_2 = \zeta + \sum_{i=1}^{m} \sigma_i (\lambda_1 c_i + \lambda_2 a_i) + \sum_{i=1}^{l} \frac{\partial \varphi}{\partial \alpha_i} e_i + \sum_{s=m+1}^{n} \vartheta_s f_s \tag{3.8}$$

Here $e_1, \ldots, e_l, f_{m+1}, \ldots, f_n$ are constants and the component ζ is a particular solution of (3.4), which by virtue of (3.7) is T-periodic in t and satisfies relations (3.6).

After an appropriate integration of these relations we obtain

$$z_r\zeta = \sum_{i=1}^{m} z_r \frac{\partial x_1}{\partial \alpha_i} a_i - \begin{cases} \lambda_1^2 a_r t & (r = 1, \ldots, m) \\ \lambda_1 c_r t & (r = m+1, \ldots, l) \\ \lambda_1 d_{r-l+m} & (r = l+1, \ldots, l+n-m) \end{cases} \tag{3.9}$$

The system of equations for determining the periodic third approximation η_3 is

$$\eta_3^{\cdot} = \left(\frac{\partial X}{\partial x}\right)\eta_3 + \left[\left(\frac{\partial'}{\partial\mu}\frac{\partial X}{\partial x}\right) - \lambda_2 E_k\right]\left[\lambda_1 \sum_{i=1}^{m} \sigma_i a_i + \sum_{i=1}^{l} \frac{\partial\varphi}{\partial\alpha_i} c_i + \sum_{s=m+1}^{n} \vartheta_s d_s\right] - \lambda_3 \sum_{i=1}^{m} \frac{\partial\varphi}{\partial\alpha_i} a_i - \lambda_1 \left[\zeta + \sum_{i=1}^{m} \sigma_i (\lambda_1 c_i + \lambda_2 a_i) + \sum_{i=1}^{l} \frac{\partial\varphi}{\partial\alpha_i} e_i + \sum_{s=m+1}^{n} \vartheta_s f_s\right]. \tag{3.10}$$

The conditions for the T-periodicity of η_3 are determined by the usual methods. Here, however, besides relations (1.14), (1.16), (2.6), (2.8)–(2.10) and (3.5) we should also keep (1.11) and (3.9) in mind. By virtue of these latter relations we have, with due regard to (3.7),

$$\int_0^T z_r \left(\frac{\partial'}{\partial\mu}\frac{\partial X}{\partial x}\right)\sigma_i dt = \sum_{s=1}^{m} \chi_{si} \frac{\partial P_r}{\partial h_s}, \qquad \int_0^T z_r \zeta dt = \sum_{i=1}^{m} p_{ri} a_i \tag{3.11}$$

$$p_{ri} = \int_0^T \left(z_r \frac{\partial x_1}{\partial \alpha_i} - \frac{\partial P_r}{\partial \alpha_i}\frac{t}{T}\right) dt$$

Finally we arrive at the following system of $l + n - m$ linear inhomogeneous equations for determining the unknowns $c_1, \ldots, c_m, e_{m+1}, \ldots, e_l, f_{m+1}, \ldots, f_n$:

$$\sum_{i=1}^{l} \frac{\partial P_r}{\partial\alpha_i} c_i + \sum_{s=m+1}^{n} \frac{\partial^* P_r}{\partial h_s} d_s + \lambda_1 \sum_{i,\, s=1}^{m} \chi_{si} \frac{\partial P_r}{\partial h_s} a_i = \lambda_1 \sum_{i=1}^{m} p_{ri} a_i + T \times \begin{cases} 2\lambda_1\lambda_2 a_r + \lambda_1^2 c_r & (r = 1, \ldots, m) \\ \lambda_2 c_r + \lambda_1 e_r & (r = m+1, \ldots, l) \\ \lambda_2 d_{r-l+m} + \lambda_1 f_{r-l+m} & (r = l+1, \ldots, l+n-m) \end{cases} \tag{3.12}$$

The construction carried out permits us to determine the first two approximations λ_1 and λ_2 to the nonsimple critical indices of the mode. Naturally, from system (3.7), the subsystem of the first m equations in the unknowns $a_1, \ldots, a_m$ arises. The condition that the latter's determinant equals zero yields the equation

$$\left|\frac{\partial P_r}{\partial\alpha_i} - \delta_{ir}\lambda_1^2 T\right|_{i,\, r=1,\, \ldots,\, m} = 0 \tag{3.13}$$

allows us to find the m values of the quantity λ_1^2. In the case of stability all these values should be real and negative. We say that the corresponding m inequalities ($\lambda_1^2 < 0$) constitute stability criteria of the second group for the periodic mode.

The fulfillment of the stability criteria of the second group ensures only that the first approximations to the nonsimple characteristic indices are imaginary ($\mathrm{Re}\,\lambda_1 = 0$). Therefore, a complete judgment on the signs of the real parts of these indices is obtained from the expressions for the second approximations λ_2. The appropriate expressions are easily obtained from the first m equations in (3.12), which form the following linear inhomogeneous system for determining the constants $c_1, \ldots, c_m$

$$\sum_{i=1}^{m} \frac{\partial P_r}{\partial \alpha_i} c_i - \lambda_1^2 T c_r = \lambda_1 \left\{ 2\lambda_2 T a_r - \sum_{i=1}^{m} \left[\sum_{s=1}^{m} \chi_{si} \frac{\partial P_r}{\partial h_s} + \frac{1}{\lambda_1^2 T} \left(\sum_{s=m+1}^{l} \frac{\partial P_r}{\partial \alpha_s} \frac{\partial P_s}{\partial \alpha_i} + \sum_{s=m+1}^{n} \frac{\partial^* P_r}{\partial h_s} \frac{\partial P_{s+l-m}}{\partial \alpha_i} \right) - p_{ri} \right] a_i \right\} \qquad (3.14)$$

In the derivation of system (3.14) the magnitudes of the constants $c_{m+1}, \ldots, c_l, d_{m+1}, \ldots d_n$ were eliminated by means of (3.7). The determinant of the homogeneous part of system (3.14) coincides with (3.13) and, consequently, vanishes. Therefore, to be able to solve this system we should impose specific constraints on its right-hand sides. The corresponding relations, in a form solved with respect to λ_2 are

$$\lambda_2 = \frac{1}{2T \sum_{r=1}^{m} a_r a_r^*} \sum_{i,\, r=1}^{m} \left[\sum_{s=1}^{m} \chi_{si} \frac{\partial P_r}{\partial h_s} + \frac{1}{\lambda_1^2 T} \left(\sum_{s=m+1}^{l} \frac{\partial P_r}{\partial \alpha_s} \frac{\partial P_s}{\partial \alpha_i} + \sum_{s=m+1}^{n} \frac{\partial^* P_r}{\partial h_s} \frac{\partial P_{s+l-m}}{\partial \alpha_i} \right) - p_{ri} \right] a_i a_r^* \qquad (3.15)$$

Here the numbers $a_1^*, \ldots, a_m^*$ form the solution of the system

$$\sum_{r=1}^{m} \frac{\partial P_r}{\partial \alpha_i} a_r^* = \lambda_1^2 T a_i^* \qquad (i = 1, \ldots, m) \qquad (3.16)$$

conjugate to the first group of Eqs. (3.7).

If all m roots λ_1^2 of Eq. (3.13) are nonzero and have simple elementary divisors (we shall assume this), then to each such root there corresponds its own set of numbers $a_1, \ldots, a_m, a_1^*, \ldots, a_m^*$ and its own magnitude of λ_2 computed by formula (3.15). Thus, the nonsimple characteristic indices of the mode separate in a natural fashion into m pairs of the form

$$\lambda_r^{(1,2)} = \pm \lambda_1^{(r)} \mu^{1/2} + \lambda_2^{(r)} \mu \pm \mu^{3/2} \ldots \qquad (r = 1, \ldots, m) \qquad (3.17)$$

If the stability criteria of the second group are fulfilled, then all the numbers λ_2 are real, and for a definitive judgment on stability we should verify the fulfillment of the m inequalities $\lambda_2^{(r)} < 0$. We call this group of inequalities the stability criteria of the third group.

In conclusion we note that the relations obtained in this paper permit us to predetermine the existence and the stability of the mode in the autonomous case as well. To do this, however, the generating system should be chosen so that the frequency of its periodic solution equals unknown frequency ν of the desired mode. Furthermore, the quantity ν in Eqs. (1.5), (1.17) should be replaced by the first approximation ν_0 to it. Then in the autonomous case the quantities $\nu_0, \alpha_2 - \alpha_1, \ldots, \alpha_l - \alpha_1, h_1, \ldots, h_n$ are uniquely determined from these equations. It is also essential here that the determinant of system (2.13) have a zero root, and the total number of stability criteria of the first group is lessened by unity. We can convince ourselves of the latter by summing the first l columns of the determinant of system (2.13) with due regard to the fact that

$$\partial P_r / \partial \alpha_1 + \ldots + \partial P_r / \partial \alpha_l \equiv 0$$

REFERENCES

1. Malkin, I. G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
2. Blekhman, I. I., On the stability of the periodic solutions of quasi-linear nonautonomous systems with many degrees of freedom. Dokl. Akad. Nauk SSSR, Vol.104, №6, 1955.
3. Chetaev, N. G., The Stability of Motion. (English translation), Pergamon Press, Book №09505, 1961.
4. Akulenko, L. D., On the oscillatory and rotational resonant motions. PMM Vol. 32, №2, 1968.
5. Nagaev, R. F., Synchronization in a system of essentially nonlinear objects with a single degree of freedom. PMM Vol. 29, №2, 1965.
6. Lur'e, A. I., Analytical Mechanics. Moscow, Fizmatgiz, 1961.
7. Kushul', M. Ia., On the quasi-harmonic systems close to systems with constant coefficients, in which the pure imaginary roots of the fundamental equation have nonsimple elementary divisors. PMM Vol. 22, №4, 1958.

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FORCED OSCILLATIONS WITH A SLIDING REGIME RANGE OF A TWO-MASS SYSTEM INTERACTING WITH A FIXED STOP

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Iu. S. FEDOSENKO
(Gor'kii)

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We investigate, by the method developed in [1], the forced oscillations with a sliding regime range of a two-mass system with elastic connection between the elements, impacting a fixed stop. The system being considered is a dynamic model for a number of vibrational mechanisms. Forced oscillations with a sliding regime range of a system with shock interactions are periodic motions accompanied by a period of an infinite succession of instantaneous collisions of two fixed elements of the model [2]. Within the framework of conditions of roughness of the parameter space [3], in this paper we study by the method of [1] periodic motions with a sliding regime range of a two-mass system with a stop. This problem was posed because in real systems the velocity recovery factor R changes from shock to shock, mainly taking small values (0, 0.2). At the same time, the regions of realizability of one-impact oscillations, in practice the most essential ones among motions with a finite number of interactions over a period, narrow down sharply as R decreases and becomes very small even for $R < 0.6$ [4]. Thus, the stability of the given operation can be ensured by a law of motion which is independent or weakly dependent on R (*) (see footnote on the next page). By virtue of what has been said above,